

Normal Mode Spectrum of the Deuteron in the Skyrme Model

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Abstract

The normal mode spectrum of the deuteron in the Skyrme model is computed. We find a bound doublet mode below the pion mass, which can be related to the well-known 90° scattering of two skyrmions. We also find a singlet ‘breather’ mode and another doublet above the pion mass. The qualitative pattern of the spectrum is similar to that recently found for the $B = 4$ multiskyrmion. The symmetries of all the vibrational modes are presented.

1 Introduction

The Skyrme model [1] has had some success in describing both single nucleon properties and the nucleon-nucleon interaction [2–4]. It is therefore of interest to see how well the model performs for larger nuclei. Classical multiskyrmion solutions are known up to baryon number $B = 9$ [5,6]. However, these must be quantised before any comparison to the real world can be made.

It is generally agreed that a proper treatment of the Skyrme model as a quantum field theory is difficult. Instead, a limited number of degrees of freedom are quantised, and usually only the zero mode collective coordinates [2]. This reduces the model to finite-dimensional quantum mechanics. The conventional wisdom is that $6N$ degrees of freedom are required to describe a system containing N nucleons (as would be required for N widely separated skyrmions).

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However, minimal energy solutions for $B > 1$ are in general single large solitons, which have a maximum of 9 zero modes. Therefore even for a very limited quantisation it is necessary to include additional discrete modes.

This is the principal motivation for the calculation of the vibrational spectra of multiskyrmions. A full quantisation should also include the effects of the soliton on the vacuum fluctuations, or radiation. We shall not attempt this here, but will merely present a new method for computing normal mode spectra, with particular reference to the $B = 2$ solution. The results for $B = 4$ are described in a concurrent paper [9]. A similar calculation was recently performed by Walet [7] for $B = 2$ and $B = 3$, using the Yang-Mills instanton-induced form for the Skyrme fields. The main differences in the present calculation is that we do not employ this approximation, and also explicitly include a pion mass term in the Lagrangian. This mass term clearly plays a key role in determining which vibrational modes are bound.

The vibrational spectra are interesting in their own right, especially since we have discovered hitherto unsuspected similarities between the two cases considered. In particular, the spectrum divides into two types of modes. The lower modes correspond to those expected from an approximate correspondence between BPS monopoles and Skyrmions [9]. Going up in frequency, one encounters first the ‘breather’ and then higher multipole breathing modes. While the structure of the spectrum is interesting, we emphasise that it represents merely a first step in a rather more ambitious long term project, namely the computation of properties of quantised multiSkyrmions.

2 Computation of the Spectrum

The basic idea behind our method for calculating the bound state spectrum is quite simple. First, the classical solution is obtained by numerical relaxation. It is then very slightly perturbed, and the resulting configuration evolved for a long time at constant energy. During this process, field values at certain points are recorded as a function of time. This data can then be Fourier analysed to obtain power spectra. Both the relaxation to the classical solution, and the subsequent time evolution after perturbation, require numerical solution of the Skyrme equations. The development of a fast computer code to perform this task lies at the heart of our method. In the current letter, we shall only briefly outline the main ideas underlying our algorithm; further details may be found in [10].

The Skyrme model has three free parameters: F_π , e and m_π . The work of Walet [7] indicates that binding energies after quantisation probably depend very sensitively on these parameters. We defer further discussion of parameters

until the full calculation has been performed. Instead, we will express our results in terms of the usual dimensionless Skyrme units [12]. In these units, the topological lower bound on the energy is $12\pi^2 B$, and the Skyrme Lagrangian density is

$$\mathcal{L} = \frac{1}{2} \text{Tr}(L_\mu L^\mu) + \frac{1}{16} \text{Tr}([L_\mu, L_\nu][L^\mu, L^\nu]) + 4\beta^2(\text{Tr}U - 2), \quad (1)$$

where $\beta = m_\pi/F_\pi e$ is a dimensionless constant, $L_\mu = U^\dagger \partial_\mu U$ and $U = \sigma + i\vec{\pi} \cdot \vec{\tau}$ is the $SU(2)$ -valued scalar field (τ_i are the Pauli matrices). For definiteness, we have set $\beta = 0.263$, following Adkins and Nappi[8], although eventually we would hope to extend the calculation to treat β as a free parameter.

The action $S = \int d^4x \mathcal{L}$ is discretised on a finite lattice, and the equations of motion are derived by varying with respect to each field component $\phi^a(i, j, k, t)$ (where $\phi = (\sigma, \vec{\pi})$), at each point in the lattice. The $SU(2)$ constraint is enforced by the introduction of a term $\lambda(\phi^2 - 1)$, where λ is a Lagrange multiplier. At this point we run into the usual problem that the “kinetic” (ie. time dependent) part of the quartic Skyrme term causes coupling between nearest neighbours in time and space. The problem is trivially solved for the initial relaxation by simply ignoring the offending terms, which are irrelevant for a static solution. But to compute the normal modes we must consider small fluctuations about the static solution: $\phi = \phi_0 + \delta\phi$. If we linearise in $\delta\phi$ (just for the problematic terms), then the system of equations uncouples, and we are left with an equation of the form

$$\phi^a(i, j, k, t+1) = \tilde{\lambda} \phi^a(i, j, k, t) + R^a(i, j, k, t) \quad (2)$$

at each point, where $\tilde{\lambda}$ is a constant related to the Lagrange multiplier. The $SU(2)$ constraint can then be invoked for $\phi(t+1)$, to produce a quadratic equation which can be solved for $\tilde{\lambda}$. This can then be substituted back into Equation (2) to define the time evolution.

3 Results and Interpretation of Spectra.

Our simulations are carried out on a finite lattice of N^3 points, of total volume L^3 , so that the lattice spacing $\Delta x = L/N$. Deuteron spectra have been calculated for a variety of box sizes and lattice spacings. We find that the frequencies observed show no discernible dependence on lattice spacing, provided that the lattice spacing is no larger than about 0.125 in our length units. There is, however, significant dependence on box size. Figure 1 shows deuteron spectra for $L = 6$ and $L = 8$. Our code uses periodic boundary conditions, so

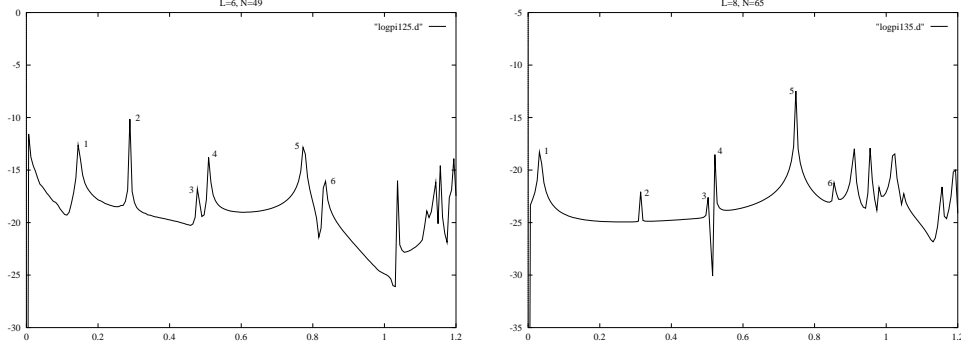


Fig. 1. Deuteron power spectra, for two different box sizes ($L=6$ and $L=8$). $\text{Log}(\text{power})$ is plotted against angular frequency ω .

the finite size effects can be understood in terms of the interaction between neighbouring ‘image’ solitons. Since a pion mass term is included explicitly in the Lagrangian, these effects should decay exponentially as L increases. The frequencies observed for $L = 8$ should already be a reasonable approximation to the infinite separation (box size) limit; however, this can eventually be improved, with more data, by fitting the exponential curve, and extrapolating.

Having obtained the normal mode spectrum, the next problem is to interpret it. It is necessary to distinguish between vibrational modes and radiation, and we also wish to determine the degeneracy and symmetry of all modes. Note that radiative modes are discrete, rather than forming a continuum above the pion mass, because of the finite box size. For the same reason, bound states are possible above the pion mass. In an infinite volume box, radiation continuum modes at the frequencies of the vibrational modes would mix with the latter, causing them to be spatially extended out to infinity. In a finite box, however, the relevant continuum modes may be absent, causing the vibrational modes above the pion mass to be spatially localised.

To gain information about the observed peaks, the process of time evolution is repeated, this time projecting out all previously identified frequencies. We thus obtain Fourier amplitudes for each mode, at all points on the lattice for each field component

$$A_{\omega}^a(i, j, k) = \int dt \delta\phi^a(i, j, k, t) \cos\omega t \quad (3)$$

where $\delta\phi^a = \phi^a - \phi_0^a$ is the difference between the actual field value at a given time, and the static solution. For each frequency, the total time of integration is chosen to be an exact multiple of the period of oscillation.

The inner product between two such amplitudes can be defined

$$\langle A_\omega | A_{\omega'} \rangle = \int d\mathbf{x}^3 K^{ab}(\vec{x}) A_\omega^a(\vec{x}) A_{\omega'}^b(\vec{x}), \quad (4)$$

with summation implied, where $K^{ab} = \delta^{ab}(1 + (\partial_i \phi_0)^2) - \partial_i \phi_0^a \partial_j \phi_0^b$ is the matrix which multiplies $\partial_0^2 \phi^b$ on the left hand side of the (linearised) Skyrme equations. It effectively plays the role of a spatially dependent inertia tensor in the dynamics. The inner product is defined in this way to ensure orthogonality between modes of different frequency.

The most immediate concern is to distinguish between bound modes and radiation in the spectrum. To do this, it is useful to calculate the norm of a given mode as a function of radius. Instead of integrating $\langle A_\omega | A_\omega \rangle$ over the whole box, we instead integrate only for N^3 points in a cube in the middle. By varying N , $\langle A_\omega | A_\omega \rangle$ is mapped out as a function of “radius” (or side length). If a particular mode is bound, then it should be localised at or near the static solution. Thus the curve of $\langle A_\omega | A_\omega \rangle$ vs N should rise steeply at a radius corresponding roughly to the soliton, then flatten out as the edge of the box is approached. Radiation, on the other hand, should be spread evenly throughout the box, giving a curve going approximately as N^3 , rising most steeply at the edge. The curves for the six modes labelled in Figure 1 are plotted in Figure 2. It is clear that the curves do indeed display characteristic shapes, so that this method successfully provides a clear distinction between bound modes and radiation. We conclude that modes 3 and 4 are radiative, whereas modes 1, 2, 5 and 6 are bound.

The next obvious question is the degeneracy of all the modes. If a given frequency ω is n -fold degenerate, then an $m \times m$ matrix of inner products $\langle A_\omega^i | A_\omega^j \rangle$ (where $m > n$), will only have n non-zero eigenvalues. Fourier amplitudes from different runs, with different random initial perturbations, (but for the same frequency), can be used to calculate the degeneracies via this method. However, cleaner results are often obtained by taking one amplitude, then generating more by applying different symmetry operations. The symmetries used must leave the static solution invariant; they generally consist of a physical transformation followed by an isospin transformation of the pion fields. Using this method, we find that modes 3 and 5 are singlets, while peaks 1, 2, 4, and 6 correspond to doublets.

The only remaining issue concerns symmetry. We wish to classify all the modes according to the representation they form of the symmetry group of the static solution. The classical $B = 2$ multiskyrmion is axially symmetric, and in addition possesses a reflection symmetry in the plane of the torus. The symmetry group is $D_{\infty h}$, axial symmetry extended by inversion. Its character table is given in Table 1. The notation of Hamermesh[11] has been used for the conju-

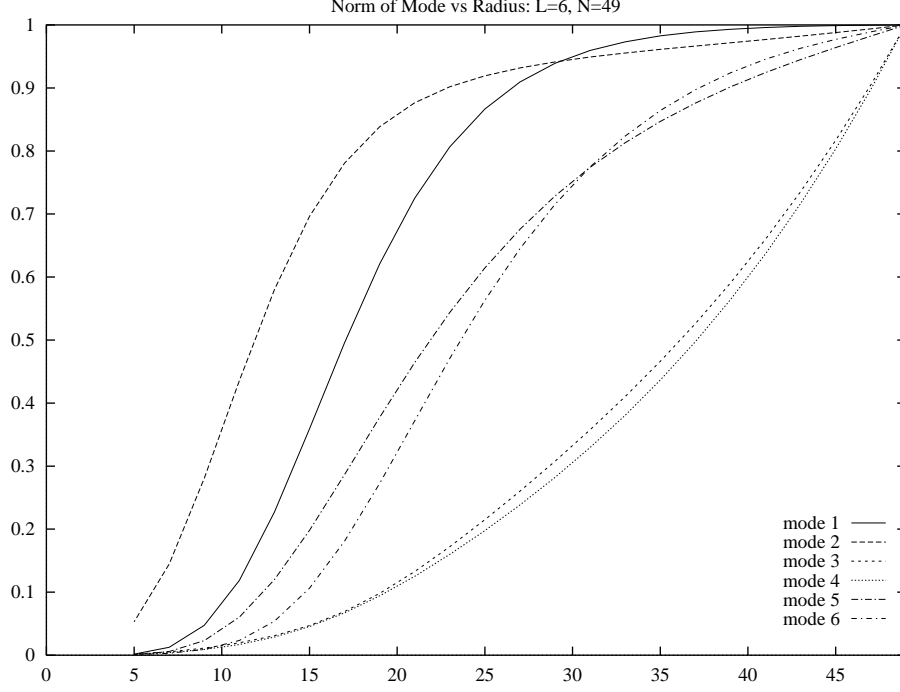


Fig. 2. $\langle A_\omega | A_\omega \rangle$ vs N , for the six lowest modes in the deuteron spectrum. All modes are normalised so that $\langle A_\omega | A_\omega \rangle = 1$ when integrated over the whole box

gacy classes and the singlet representations. E is the identity, $C(\phi)$ is rotation by angle ϕ , σ_v is reflection in any plane including the z -axis, and I is inversion. The singlet representations are labelled g or u to indicate whether they are even or odd under parity, while the $+$ or $-$ refers to their behaviour under σ_v . There are an infinite number of doublets, labelled by an integer n , and parity.

It is clear that singlet representations are distinguished by their properties under σ_v and I . The symmetries of modes 3 and 5 can therefore be determined by taking matrix elements of these operators

	E	$C(\phi)$	σ_v	I	$IC(\phi)$	$I\sigma_v$
Σ_g^+	1	1	1	1	1	1
Σ_u^+	1	1	1	-1	-1	-1
Σ_g^-	1	1	-1	1	1	-1
Σ_u^-	1	1	-1	-1	-1	1
n^+	2	$2 \cos n\phi$	0	2	$2 \cos n\phi$	0
n^-	2	$2 \cos n\phi$	0	-2	$2 \cos n\phi$	0

Table 1
Character table for $D_{\infty h}$

$$\begin{aligned}
\langle 3|\sigma_v|3\rangle &= 1 & \langle 3|I|3\rangle &= -1 \\
\langle 5|\sigma_v|5\rangle &= 1 & \langle 5|I|5\rangle &= 1.
\end{aligned} \tag{5}$$

For doublet states, the simplest way to determine n is to search for the lowest angle ϕ such that $\langle X|C(\phi)|X\rangle = 0$ (this implies that $\langle X|C(2\phi)|X\rangle = -1$ and $\langle X|C(4\phi)|X\rangle = 1$). If the lowest angle $\phi = 90^\circ$ then $n = 1$, if $\phi = 45^\circ$ then $n = 2$, and so on. In practice, $n = 1$ or 2 are the only realistic values, so the matrix elements of $C(45^\circ)$ and $C(90^\circ)$, together with inversion, provide sufficient information to distinguish the representations.

$$\begin{aligned}
\langle 1|C(45^\circ)|1\rangle &= \frac{1}{\sqrt{2}} & \langle 1|C(90^\circ)|1\rangle &= 0 & \langle 1|I|1\rangle &= 1 \\
\langle 2|C(45^\circ)|2\rangle &= 0 & \langle 2|C(90^\circ)|2\rangle &= -1 & \langle 2|I|2\rangle &= 1 \\
\langle 4|C(45^\circ)|4\rangle &= 0 & \langle 4|C(90^\circ)|4\rangle &= -1 & \langle 4|I|4\rangle &= 1 \\
\langle 6|C(45^\circ)|6\rangle &= \frac{1}{\sqrt{2}} & \langle 6|C(90^\circ)|6\rangle &= 0 & \langle 6|I|6\rangle &= -1
\end{aligned} \tag{6}$$

To summarise, the first peak is a doublet, with symmetry 1^+ . This is the representation under which axial vectors transform in $D_{\infty h}$. These states do not correspond to true vibrations, but rather represent two broken zero modes, spatial rotations of the torus around the x and y axes. The periodic boundary conditions of the lattice introduce the possibility of low energy ‘spin waves’, whose frequency will decrease as the box size increases, eventually reverting to zero modes in the infinite limit. This explains the extreme shift in frequency observed for this mode when the box size is changed from $L = 6$ to $L = 8$.

The second peak in the spectrum represents the first true finite frequency bound modes. They are doubly degenerate, with a quadrupole symmetry 2^+ . If amplified, these would become the classic 90° scattering of two skyrmions. There is also a close analogy to the scattering of BPS monopoles.

The zero modes of the two-monopole toroidal BPS solution may be fairly simply understood. The BPS solutions are associated with rational maps constructed as follows. A complex variable z is associated with each direction in space through a projective map. The complex z plane is embedded in three Euclidean dimensions, and a unit two-sphere is drawn centered on the origin. Each direction \vec{n} defines a point on the two-sphere, which is then projected from one of the poles of the two-sphere onto the z plane. One then defines a rational map R which is a ratio of polynomials in z , each up to z^N for a N -monopole solution. Just as z defines a direction in real space, R defines one in internal space, acted on by the global $SU(2)$ symmetry via Moebius transformations $R \rightarrow (\alpha R + \beta)/(-\beta^* R + \alpha^*)$, with α and β being the 11 and 12 matrix elements of the $SU(2)$ matrix.

For special choices of R , the solutions exhibit enlarged symmetry. In our case, the 2-monopole solution represented by $R = z^2$ is the toriodal solution which has $D_{\infty h}$ symmetry. It is invariant under rotations $z \rightarrow e^{i\phi}z$ combined with isospin rotations $R \rightarrow e^{-2i\phi}R$, and also inversions $z \rightarrow -1/z^*$, $R \rightarrow 1/R^*$. The general perturbed 2-monopole solution is represented by $R = (z^2 + az + b)/(cz^2 + dz + 1 + e)$ where the coefficient of z^2 in the numerator is taken to be unity and the complex parameters a, b, c, d, e are small. These parameters are not invariant under $D_{\infty h}$, but transform according to some irreducible representation, which has ten real dimensions. It is straightforward to check by computing the characters of the symmetry group elements that the representation is $2^+ + 2^- + 1^+ + \Sigma_g^- + 1^- + \Sigma_u^-$. The isospin zero modes comprise the 2^- representation, the rotational zero modes comprise the $1^+ + \Sigma_g^-$ representation and the translational zero modes the $1^- + \Sigma_u^-$ representation. This leaves the 2^+ representation as the remaining nontrivial zero mode of the 2-monopole BPS solution. And this representation coincides with our results for the lowest vibrational mode of the deuteron in the Skyrme model.

The next two frequencies correspond to radiation, with wave number $k = 0$. These modes are split, rather than forming a degenerate triplet, because the pion fields of the static solution do not all transform the same way. The radiative modes fall into the same representations as the static solution pions: $\Sigma_u^+ + 2^+$.

The fifth peak is a breathing mode, where the size of the soliton fluctuates with no change in shape. It is a singlet, and transforms trivially under $D_{\infty h}$, as would be expected. The sixth peak represents the last bound states. These form a doublet, with symmetry 1^- . The torus expands on one side while contracting on the other, a dipole ‘breathing’ motion. Altogether then, we have found five finite energy normal modes in the deuteron spectrum.

4 Conclusions.

The normal mode spectrum of the deuteron has been calculated. The nature of all states has been identified, and their degeneracies and symmetries calculated. We find a bound doublet below the pion mass, which is related to 90° scattering between two skyrmions. Two more bound states appear above the pion mass: a breather, plus a second doublet with a dipole symmetry.

The overall pattern of the deuteron spectrum displays a remarkable similarity to that recently found for the $B = 4$ solution [9]. There again, all modes below the pion mass can be related to those expected from an approximate correspondence with the zero modes of BPS multimonopoles. The first $B = 4$

bound mode above the pion mass is a breather, and the remaining bound modes involve ‘breathing’ motion of the baryon density. It is tempting to speculate that a similar pattern will be observed for all multiskyrmion spectra, but more examples are required to confirm this.

One further remark may be worthwhile. We have found five finite energy bound modes in the deuteron spectrum. Added to the eight zero modes, this makes a total of 13. Sixteen bound modes were found for $B = 4$, giving a total of 25. In both cases, then, there is a total of $6B + 1$ modes, one more than is usually expected. The “extra” mode is the breather; every multiskyrmion, including the single skyrmion, must have such a mode. The remaining vibrational modes can be considered as corresponding to the broken zero modes of the separated skyrmions.

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Fig. 3. Deuteron vibrational modes: plots of $\mathcal{B}(\phi_0 + \delta\phi) - \mathcal{B}(\phi_0)$. Isosurfaces indicate surfaces of constant negative perturbation; contours positive. Note that all modes except mode 3 are plotted looking more or less directly down the z -axis, so that whereas modes 2 and 4 are quadrupoles in the (x, y) -plane, mode 1 is a dipole, but is odd under reflections in this plane. Mode 3 is odd under z -reflections, but the twisting is merely an artifact due to slight contamination from mode 2.

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